Scaling Behavior in the Einstein-Yang-Mills Monopoles and Dyons

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Abstract

Scaling behavior in the moduli space of monopole and dyon solutions in the Einstein-Yang-Mills theory in the asymptotically anti-de Sitter space is derived. The mass of monopoles and dyons scales with respect to their magnetic and electric charges, independent of the values of the cosmological constant and gauge coupling constant. The stable monopole and dyon solutions are approximated by solutions in the fixed anti-de Sitter spacetime. Unstable solutions can be viewed as the Bartnik-McKinnon solutions dressed with monopole and dyon solutions in the fixed anti-de Sitter space.

1 Introduction

It has been shown that there exist a continuum of stable and unstable monopole and dyon solutions in the Einstein-Yang-Mills theory in the asymptotically anti-de Sitter (AdS) space.[1, 2] They generalize a discrete family of unstable particle-like solutions in the asymptotically Minkowski or de Sitter space.[3] Similarly, black hole solutions exist with discrete values of magnetic charges in the asymptotically flat or de Sitter space,[4, 5] and with continuous values of non-abelian electric and magnetic charges in the asymptotically AdS space.[6, 2, 7, 8]

Monopole and dyon solutions are characterized by their mass and non-Abelian magnetic/electric charges. The spectrum defines the moduli space of the solutions, which varies with the consmological constant (Λ) and the gauge and gravitational coupling constants (e and G). The spectrum consists of infinitely many discrete points for $\Lambda \geq 0$, whereas it has a finite number of continuous branches for $\Lambda < 0$. When the parameter $\Lambda < 0$ approaches zero, an already-exsiting branch of monopole and dyon solutions collapses to a single point in the moduli space. At the same time new branches of solutions emerge. A fractal structure in the moduli space has been observed.[2, 9]

In this paper we derive a scaling law for the mass spectrum of the solutions with respect to their magnetic and electric charges (Q_M and Q_E), the cosmological constant $\Lambda(< 0)$, and the ratio of the gravitational constant to the gauge coupling constant $v \equiv 4\pi G/e^2$. Some of the results in [2] indicate that the mass of monopoles and dyons is expressed in terms of a universal function $f(Q_M, Q_E)$. We shall show that this follows from the factorization property of the solutions and that $f(Q_M, Q_E)$ is determined by the monopole and dyon solutions in the fixed AdS background metric.

AdS spacetime has many special properties. In some models it accommodates the holographic principle; the information on the boundary of the space determines physics in the bulk.[10] We shall see a trace of this property in the classical Einstein-Yang-Mills theory. The existence of stable monopole and dyon solutions in the asymptotically AdS space seems tightly connected to boundary data on non-Abelian charges, though more thorough investigation is necessary.

2 Monopoles and dyons

There exist static, spherically symmetric monopole and dyon solutions in the Einstein-Yang-Mills theory. The action of the system is

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G}(R - 2\Lambda) - \frac{1}{4}F^{a\mu\nu}F^a_{\ \mu\nu} \right].\tag{1}$$

The Einstein and Yang-Mills equations are given by

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}(R - 2\Lambda) = 8\pi G\ T^{\mu\nu}$$
$$F^{\mu\nu}_{\ \ ;\mu} + e[A_\mu, F^{\mu\nu}] = 0\ \ .\tag{2}$$

The metric of spacetime is given by

$$ds^2 = -\frac{H}{p^2}\,dt^2 + \frac{dr^2}{H} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$
$$H = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\tag{3}$$

where H, p, and m depend on r only. $m(r)/G$ represents a total mass contained inside r in the $c = 1$ unit. (Numerical values below are given in the $c = \hbar = G = 1$ unit.) The $SU(2)$ Yang-Mills fields take

$$A^{(0)} = \frac{\tau^j}{2e}\left\{ u(r)\frac{x_j}{r}dt - \epsilon_{jkl}\frac{1 - w(r)}{r^2}x_k dx_l \right\}\tag{4}$$

in the Cartesian coordinates ($x_1^2 + x_2^2 + x_3^2 = r^2$). The gauge coupling constant is denoted as e. With these ansatz (3) and (4) the Einstein and Yang-Mills equations (2) reduce to

$$\left(\frac{H}{p}w'\right)' = -\frac{p}{H}u^2w - \frac{w\,(1 - w^2)}{p}\frac{1}{r^2}\tag{5}$$

$$\left(r^2pu'\right)' = \frac{2p}{H}w^2u\tag{6}$$

$$m' = v\left[H(w')^2 + \frac{(1 - w^2)^2}{2r^2} + \frac{1}{2}r^2p^2(u')^2 + \frac{u^2w^2p^2}{H}\right]\tag{7}$$

$$p' = -\frac{2v}{r}p\left[(w')^2 + \frac{u^2w^2p^2}{H^2}\right]\tag{8}$$

with the boundary conditions $u = m = 0$ and $w = p = 1$ at the origin. The set of the equations contains two parameters, the cosmological constant Λ and the ratio of the gravitational constant to the gauge coupling constant $v = 4\pi G/e^2$.

There are soliton-type solutions with finite masses. There are infinitely many conserved, gauge-covariant charges. In the spherically symmetric case the nonvanishing charges of importance are [1, 2]

$$
\begin{pmatrix} Q_E \\ Q_M \end{pmatrix} = \frac{e}{4\pi} \int dS_k \sqrt{-g} \, \mathrm{Tr} \left(\frac{F^{k0}}{\tilde{F}^{k0}} \right) \frac{x^j \tau^j}{r}
$$
$$
= \begin{pmatrix} -u_1 p_\infty \\ 1 - w_\infty^2 \end{pmatrix} \, . \tag{9}
$$

where u_1, p_∞, and w_∞ are defined by the asymptotic expansion $u \sim u_\infty + (u_1/r) + \cdots$ etc. Each solution is specified by its mass (multiplied by G), $M = m(\infty)$, non-Abelian electric and magnetic charges, Q_E and Q_M, and the number, n, of the nodes of $w(r)$. For $\Lambda \geq 0$ the spectrum of the solutions is discrete, $u(r) = 0$ ($Q_E = 0$), $n = 1, 2, 3, \cdots$, and all solutions are unstable. For $\Lambda < 0$ the spectrum is completely different. It is continuous. For each n ($= 0, 1, 2, \cdots$) there are a family of solutions with continuous values of Q_E and Q_M.[1, 2] In particular, the nodeless solutions ($n = 0$) are stable. In the moduli space of the solutions, M of a particular point (solution) is a function of Λ, v, n, Q_E, and Q_M. M, Λ, and v have dimensions of (length), (length)$^{-2}$, and (length)^2, respectively, whereas n, Q_E, and Q_M are dimensionless. We show that M is expressed in terms of a universal function of Q_E and Q_M up to an overall factor.

3   Solutions in the fixed AdS background metric

To understand why stable solutions exist only in the asymptotically AdS space ($\Lambda < 0$), we consider soliton solutions in the fixed AdS background metric, setting $p = 1$ and $H = 1 - \Lambda r^2/3$ in Eqs. (5) and (6) and on the r.h.s. of Eq. (7). Introduce $x = (|\Lambda|/3)^{1/2} r$ and $\hat{u} = (3/|\Lambda|)^{1/2} u$. Then Eqs. (5), (6), and (7) become

$$
\frac{d}{dx} \left\{ (1 + x^2) \frac{dw}{dx} \right\} = -\frac{w(1 - w^2)}{x^2} - \frac{\hat{u}^2 w}{1 + x^2}
$$
$$
\frac{d}{dx} \left\{ x^2 \frac{d\hat{u}}{dx} \right\} = \frac{2w^2 \hat{u}}{1 + x^2} \tag{10}
$$

and

$$
\frac{dm}{dx} = v \sqrt{\frac{|\Lambda|}{3}} \left\{ (1 + x^2) \left(\frac{dw}{dx} \right)^2 + \frac{(1 - w^2)^2}{2x^2} + \frac{x^2}{2} \left(\frac{d\hat{u}}{dx} \right)^2 + \frac{\hat{u}^2 w^2}{1 + x^2} \right\} . \tag{11}
$$

The equations for $\hat{u}(x)$ and $w(x)$ do not involve either v or Λ. The charges of the solutions are

$$Q_M = 1 - w_\infty^2 \quad , \quad Q_E = x^2 \frac{d\hat{u}}{dx}\bigg|_{x=\infty} \quad . \tag{12}$$

Hence a family of the solutions in the fixed AdS background metric satisfy

$$
\begin{aligned}
w(r; \Lambda, w_\infty, Q_E)^{\text{AdS}} &= \tilde{w}^{\text{AdS}}(x; w_\infty, Q_E) \\
u(r; \Lambda, w_\infty, Q_E)^{\text{AdS}} &= \sqrt{\frac{|\Lambda|}{3}} \tilde{u}^{\text{AdS}}(x; w_\infty, Q_E) \quad .
\end{aligned}
\tag{13}
$$

Here $\{\tilde{w}^{\text{AdS}}(r), \tilde{u}^{\text{AdS}}(r)\}$ represents a solution for $\Lambda = -3$. Further dm/dx is expressed in terms of $\hat{u}$ and w with an overall factor $v(|\Lambda|/3)^{1/2}$, which implies that

$$
\begin{aligned}
m(r; \Lambda, v, w_\infty, Q_E)^{\text{AdS}} &= v\sqrt{\frac{|\Lambda|}{3}} \tilde{m}^{\text{AdS}}(x; w_\infty, Q_E) \\
M^{\text{AdS}} = m^{\text{AdS}}|_{r=\infty} &= v\sqrt{|\Lambda|} f(Q_M, Q_E) \quad ,
\end{aligned}
\tag{14}
$$

where $\tilde{m}^{\text{AdS}}(r)$ is the mass function for $v = 1$ and $\Lambda = -3$. $f(Q_M, Q_E)$ defines a universal scaling function as we shall see below. Note that f is a double-valued function of Q_M as $Q_M = 1 - w_\infty^2$.

The size of the solutions also scales. One definition of the size, ℓ, of a solution is given in terms of $m(r)$ by $m(\ell) = 0.5 \cdot m(\infty)$, where we have arbitrarily taken a size-factor 0.5. It immediately follows

$$\ell^{\text{AdS}} = \frac{1}{\sqrt{|\Lambda|}} h(Q_M, Q_E) \quad . \tag{15}$$

Typical solutions are depicted in fig. 1. Both $w(x)$ and $\hat{u}(x)$ monotonically decrease or increase. The solutions have at most one node in $w(r)$. The most of the energy of each solution is localized in $x < 10$.

There is a special solution[12]

$$\hat{u} = 0 \quad , \quad w = \frac{1}{\sqrt{1 + x^2}} \tag{16}$$

for which $Q_E = 0$, $Q_M = 1$, and $M = (\sqrt{3}\pi/8) v|\Lambda|^{1/2}$. $Q_M = 1$ corresponds to the same quantized magnetic charge as for the 't Hooft and Polyakov monopole.

Further, $w^{\text{AdS}} \sim 1$ and $m^{\text{AdS}} \sim 0$ for $x < 0.1$. It is also numerically confirmed that

$$\text{Max}_r \frac{2m^{\text{AdS}}}{r} \cdot \frac{1}{1 - (\Lambda r^2/3)} \sim \begin{cases} 0.03(v|\Lambda|)^{1/2} M^{1/2}|\Lambda|^{1/4} & \text{for } w_\infty > 1 \\ 0.1 M|\Lambda|^{1/2} & \text{for } w_\infty < 1 \end{cases} \tag{17}$$
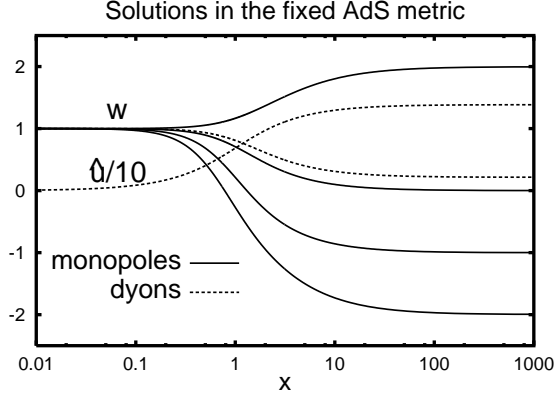
Figure 1: $w(x)$ and $\hat{u}(x)$ of typical monopole and dyon solutions in the fixed AdS metric. The particular dyon solution displayed in the figure has $(Q_M, Q_E) = (0.954, 0.527)$.

for monopole solutions. As far as $v|\Lambda|$ and $M|\Lambda|^{1/2}$ are small enough, corrections to the metric may be ignored, and the solution in the fixed AdS background metric gives a good approximation to a solution in the EYM theory. ($p(r) \sim 1$ for those solutions.)

4   Factorization

Let us turn to the EYM solutions in the $\Lambda = 0$ case. Set $u = 0$. Expressed in terms of $y = r/\sqrt{v}$, $H = 1 - (2\bar{m}/y)$ and $\bar{m} = m/\sqrt{v}$, Eqs. (5), (7), and (8) contain no parameter;

$$\frac{d}{dy}\left(\frac{H}{p}\frac{dw}{dy}\right) = -\frac{w}{p}\frac{(1-w^2)}{y^2}$$

$$\frac{d\bar{m}}{dy} = \frac{(1-w^2)^2}{2y^2} + H\left(\frac{dw}{dy}\right)^2$$

$$\frac{dp}{dy} = -\frac{2p}{y}\left(\frac{dw}{dy}\right)^2 \tag{18}$$

Solutions $\{w, p, \bar{m}\}$ are functions of y only. In each solution $w(r)$ crosses the axis n times ($n = 1, 2, \cdots$), and approaches $(-1)^n$ asymptotically. A physical mass is given by M/G, or $\bar{m}(\infty)\sqrt{v}/G = \bar{m}(\infty)M_{\text{Pl}}/\sqrt{\alpha}$ where $\alpha = e^2/4\pi$ and $G = M_{\text{Pl}}^{-2}$. The mass of the n-th Bartnik-McKinnon solution is

$$(\text{mass})_n^{\Lambda=0}(v) = \frac{M_{\text{Pl}}}{\sqrt{\alpha}} e_n \qquad (n = 1, 2, \cdots). \tag{19}$$

$e_n = M_n^{\Lambda=0}|_{v=1}$ is numerically given by $(e_1, e_2, \cdots) = (0.8286, 0.9713, \cdots)$. For $n \gg 1$, $e_n \sim 1 - 1.081 e^{-\pi n/\sqrt{3}}$.[11] $w_n^{\Lambda=0}(r)$, $p_n^{\Lambda=0}(r)$, and $m_n^{\Lambda=0}(r)$ of the n-th solution reach their

6

asymptotic values at $r \sim a_n \sqrt{v}$ where $a_n \sim 10^n$.[5] The size of the Bartnik-McKinnon solutions is characterized by $\ell_n^{\Lambda=0} \sim a_n \sqrt{v}$.

Monopole and dyon solutions in $\Lambda < 0$ are labeled by $(n, v, \Lambda, w_\infty, Q_E)$. The index n runs over 0, 1, 2, $\cdots$. We would like to show that for $\ell_n^{\Lambda=0} \sqrt{|\Lambda|} \ll 1$, the Einstein-Yang-Mills monopole solutions are well approximated by

$$
\begin{aligned}
w_n &= w_n^{\Lambda=0}(r; v) \, w^{\text{AdS}}(r; \Lambda, (-1)^n w_\infty, Q_E) \, , \\
u_n &= u^{\text{AdS}}(r; \Lambda, (-1)^n w_\infty, Q_E)/p_n^{\Lambda=0}(\infty; v) \, , \\
p_n &= p_n^{\Lambda=0}(r; v) \, , \\
m_n &= m_n^{\Lambda=0}(r; v) + m^{\text{AdS}}(r; \Lambda, (-1)^n w_\infty, Q_E, v) \, ,
\end{aligned}
\tag{20}
$$

where it has been understood that $w_0^{\Lambda=0}(r; v) = p_0^{\Lambda=0}(r; v) = 1$ and $m_0^{\Lambda=0}(r; v) = 0$. First, the solution in the fixed AdS metric approximately solves the EYM equations for $n = 0$, as remarked above.

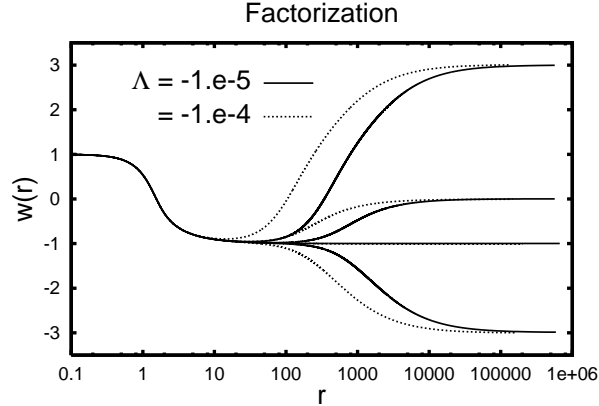

Figure 2: Factorization property of the EYM monopole solutions. $\Lambda = -10^{-4}$ and -10^{-5} with $v = 1$.

Secondly, for $n \geq 1$ we consider two regions; [I] $r < 0.1(3/|\Lambda|)^{1/2}$ and [II] $r > a_n \sqrt{v}$. The two regions overlap with each other if $|\Lambda| v < 0.03 \, a_n^{-2}$. In the region I, $w^{\text{AdS}} \sim 1$, $m^{\text{AdS}} \sim 0$, and $-\Lambda r^2/3 \ll 1$ so that the solution is well approximated by that in the $\Lambda = 0$ case, provided u^2 is sufficiently small. In the region II, $w_n^{\Lambda=0} \sim (-1)^n$, $p_n^{\Lambda=0} \sim p_n^{\Lambda=0}|_{r=\infty}$, and $m_n^{\Lambda=0} \sim \sqrt{v} e_n$. In Eqs. (5) - (8) the value of constant p is irrelevant with pu substituted by u. H is approximated by $H = 1 - \Lambda r^2/3$ as $m_n^{\Lambda=0}/r < 1/a_n \ll 1$. Hence the solutions are given by those in the fixed AdS background metric with w at $r = \infty$ given by $(-1)^n w_\infty$.

7

The solutions in the asymptotically AdS space are obtained by dressing solutions in the fixed-AdS background metric to the Bartnik-McKinnon solutions in the asymptotically flat space, as expressed in (20).

The factorization property of the solutions, (20), is confirmed by numerical evaluation of the solutions. In fig. 2, $w(r)$ of the monopole solutions at $\Lambda = -10^{-4}, -10^{-5}$ and $v = 1$ with various $w_\infty = 3, 0, -1, -3$ are depicted. For $r < 10$ these solutions are well approximated by the first Bartnik-McKinnon solution at $\Lambda = 0$. For larger r the solutions are essentially given by $-w^{\mathrm{AdS}}(x; \Lambda = -1, -w_\infty)$.

5   Scaling

A scaling law follows from the factorization property. From (14) and (20)

$$\frac{M_n(v, \Lambda, w_\infty, Q_E) - \sqrt{v}e_n}{v\sqrt{|\Lambda|}} = f(Q_M, Q_E) \tag{21}$$

the r.h.s. of which is independent of Λ and v, and also of n. The scaling law is valid for $|\Lambda|v < 0.03 a_n^{-2}$ and small $|Q_E|$.

In fig. 3 numerical data of monopole solutions for the lowest ($n = 0$) and second ($n = 1$) branches is depicted. It is seen that all data for $v|\Lambda| < 0.01$ falls on the universal function $f(Q_M, 0)$ for $n = 0$, and for $v|\Lambda| < 0.0001$ for $n = 1$. It follows from (21) that the mass is given by

$$(\mathrm{mass})_n(v, \Lambda, w_\infty, Q_E) = \frac{e_n}{\sqrt{\alpha}} M_{\mathrm{Pl}} + \frac{\sqrt{|\Lambda|}}{\alpha} f(Q_M, Q_E) \quad . \tag{22}$$

In the lowest branch the magnitude of the mass is determined by $\sqrt{|\Lambda|}/\alpha$, whereas in the higher branches it is given by $M_{\mathrm{Pl}}/\sqrt{\alpha}$.

The size of a monopole or dyon is essentially the same as that of the solution in the fixed AdS background metric, as the dressed fields cover the inside Bartnik-McKinnon core. $h(Q_M, 0)$ in (15) is depicted in fig. 4.

As $\Lambda(< 0)$ approaches 0, the branch in the Q_M-M plane collapses to a flat line $M = \sqrt{v}e_n$. The size of the solutions grows as $|\Lambda|^{-1/2}$ so that Bartnik-McKinnon solutions with higher n can be accommodated inside the AdS solutions, allowing more solutions in higher branches. This explains the phenomenon observed in ref. [2]. In the $\Lambda = 0$ limit only solutions with $Q_M = 0$ survive.
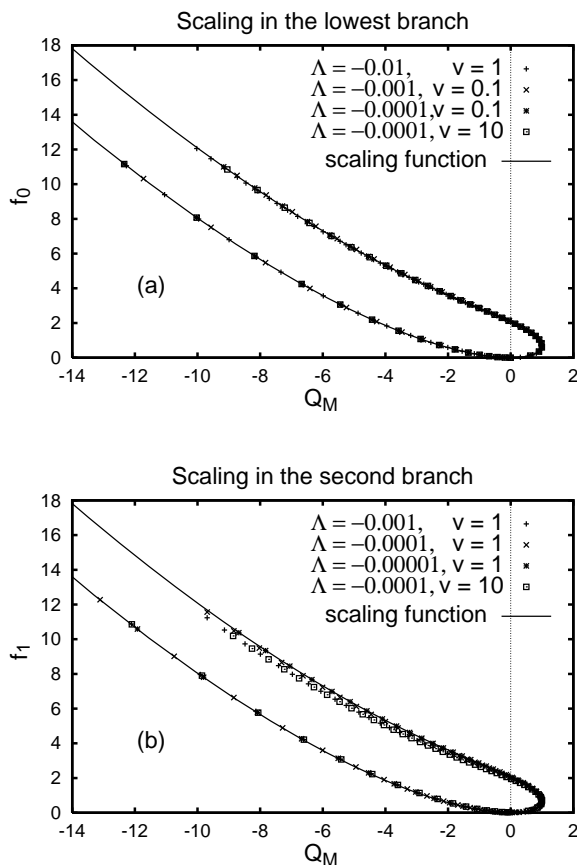
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Figure 3: Scaling (a) for $f_0 \equiv M_0(v, \Lambda, w_\infty, 0)/(v|\Lambda|^{1/2})$, and (b) for $f_1 \equiv (M_1(v, \Lambda, w_\infty, 0) - v^{1/2}e_1)/(v|\Lambda|^{1/2})$. The scaling function $f(Q_M, 0)$ is also depicted.

6   Summary

In this paper we have examined monopole-dyon solutions in the Einstein-Yang-Mills theory in the asymptotically AdS space. The monopole and dyon solutions in the lowest branch $(n = 0)$ are essentially the solutions in the fixed AdS background metric. The solutions in the higher branches $(n > 0)$ are obtained by dressing monopole and dyon solutions in the fixed AdS background metric around the Bartnik-McKinnon solutions in the asymptotically flat space. As all Bartnik-McKinnon solutions are unstable, the monopole and dyon solutions in the higher branches are unstable, whereas the nodeless solutions are stable against small perturbations.

Because of the factorization property of the solutions there arises a scaling law in the mass of the solutions when regarded as a function of e, G, Λ, Q_M and Q_E. Up to an
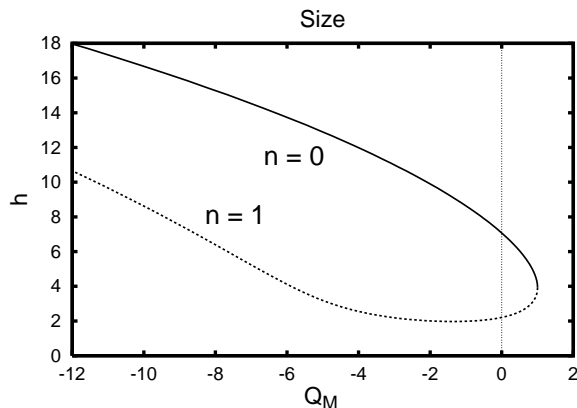
9

Figure 4: Scaling function, $h(Q_M, 0)$, for the size in (9). The portion with $n = 0$ or $n = 1$ corresponds to solutions with no node or one node in $w^{\text{AdS}}(x)$, respectively.

overall factor it scales to a universal function $f(Q_M, Q_E)$ determined by the solutions in the fixed AdS metric. The factorization/dressing mechanism is expected to apply for black hole solutions as well.

In quantum theory Q_M and Q_E are expected to be quantized. Solutions with minimal $|Q_M|$ or $|Q_E|$ must be absolutely stable. The stable solutions discussed in the present paper are, in nature, non-topological solitons. They exist only with gravitational force. In this sense they may be called gravitational solitons.[13]

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